Math 102

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October 25, 2018

Announcements

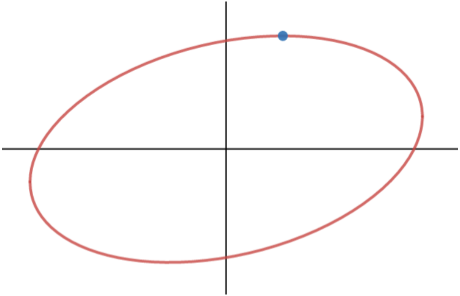
- Midterm is today/tonight! **If you do not yet know your room assignment, see me after class and I can tell you where to go.**
- Midterm policies
  - You will not be allowed to leave the room until the end of the 90 minutes, and only one student will be allowed to the washroom at a time. Make sure you plan for this. (ex. food/water)
  - No calculators or notes are allowed.
  - Bring your ID.

Goals Today

- Implicit differentiation - practice
- Exponential functions
  - Exponential growth and decay
  - The derivative
  - The natural logarithm

The Derivative of an Implicit Function

Question: Shown is the graph of $x^2 + 3y^2 - xy - 11 = 0$. The blue point is the highest point on the graph. How would we calculate its coordinates?



Strategy: At the point shown, $\frac{dy}{dx} = 0$, so let's calculate $\frac{dy}{dx}$ and set it equal to zero.

$$x^2 + 3y^2 - xy - 11 = 0$$

$$2x + 6y\frac{dy}{dx} - (x\frac{dy}{dx} + y) = 0$$

$$(6y - x)\frac{dy}{dx} + (2x - y) = 0$$

$$\boxed{\frac{dy}{dx} = \frac{y - 2x}{6y - x}}$$

$\frac{dy}{dx} = \frac{y-2x}{6y-x} = 0 \implies y = 2x$. Plug this back into our original equation $x^2 + 3y^2 - xy - 11 = 0$:

$$x^2 + 3(2x)^2 - x(2x) - 11 = 0$$

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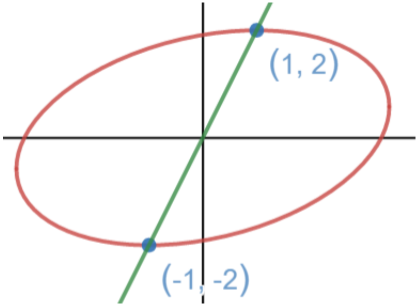
$$11x^2 - 11 = 0 \implies x = 1, y = 2 \text{ or } x = -1, y = -2$$



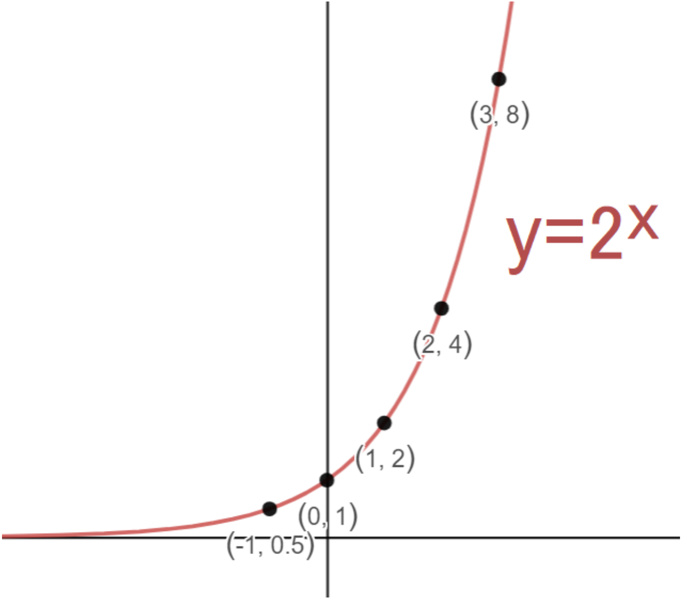
Question: Consider the same implicit function, $x^2 + 3y^2 - xy - 11 = 0$.

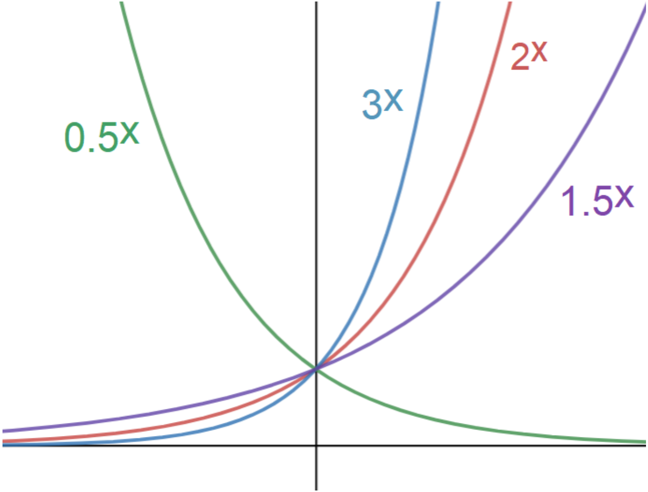
- How would you calculate the **leftmost** point on the graph?
- Suppose that x and y both depend on t. At the point $(-\frac{2\sqrt{11}}{3}, \frac{\sqrt{11}}{3})$, suppose that $\frac{dx}{dt} = 2$. Calculate $\frac{dy}{dt}$.

Exponential Functions

x	1	2	3	4	5	$\cdots$	10	$\cdots$	20	$\cdots$
x^2	1	4	9	16	25	$\cdots$	100	$\cdots$	400	$\cdots$
2^x	2	4	8	16	32	$\cdots$	1,024	$\cdots$	1,048,576	$\cdots$

- A power function has the form $f(x) = Cx^a$ for constants C and a. Examples: $x, x^2, x^3, x^{0.5} = \sqrt{x}$. a is the **exponent.**
- An exponential function has the form $f(x) = Ca^x$ for constants C and a with $a > 0$. Examples: $2^x, 3^x, 0.5^x$. a is the **base.**


(3, 8)
y=2^x
(2, 4)
(1, 2)
(0, 1)
(-1, 0.5)



- $a > 1 \implies$ **exponential growth**.
- $a < 1 \implies$ **exponential decay**.

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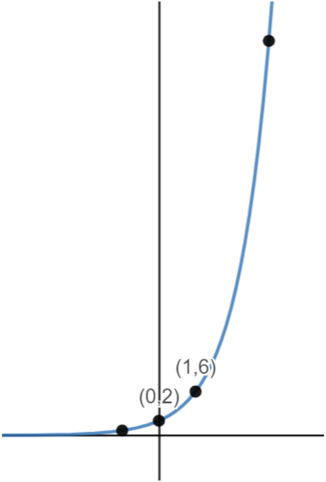

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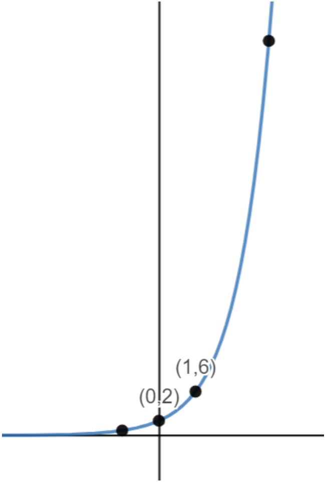

$Ca^0 = 2$ and $Ca^1 = 6$. Thus, $C = 2$ and $a = 3$, i.e. $\boxed{f(x) = 2 \cdot 3^x}$.

$f(3) = 2 \cdot 3^3 = 54$

$f(-1) = 2 \cdot 3^{-1} = 2/3$

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Question: Can we prove this? And what is C?

The derivative of $f(x) = 2^x$

By definition,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h} \\
&= \lim_{h\to 0} \frac{2^x \cdot 2^h - 2^x}{h} \\
&= \lim_{h\to 0} \frac{2^x(2^h - 1)}{h} \\
&= 2^x \left(\lim_{h\to 0} \frac{2^h - 1}{h} \right)
\end{aligned}
$$

The derivative of $f(x) = a^x$

Let $f(x) = a^x$. Then

$$f'(x) = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$$

where $\lim_{h \to 0} \frac{a^h - 1}{h}$ is a **constant that depends only on** a**.**

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where $\lim_{h \to 0} \frac{a^h - 1}{h}$ is a **constant that depends only on a.**

This constant is known as the **natural logarithm of a**, and is also denoted by $\ln(a)$.

$$\ln(a) := \lim_{h \to 0} \frac{a^h - 1}{h}$$

$$\ln(0.5) = \lim_{h\to 0} \frac{0.5^h - 1}{h} \approx -0.692147...$$

$$\ln(1) = \lim_{h\to 0} \frac{1^h - 1}{h} = 0$$

$$\ln(2) = \lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.692147...$$

$$\ln(3) = \lim_{h\to 0} \frac{3^h - 1}{h} \approx 1.098612$$

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e is the unique number such that $\ln(e) = 1$. It is between 2 and 3, and is approximately

$$e \approx 2.7182818\ldots$$

Exponential and Logarithm

Another perspective: e^x and $\ln(x)$ are **inverse functions.** That is, $e^{\ln(a)} = a$ and $\ln(e^a) = a$.

Proof of $e^{\ln(a)} = a$: Let $f(x) = (e^{\ln(a)})^x$.

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$$f'(x) = e^{\ln(a)\cdot x} \cdot \ln(a)$$

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It therefore follows that $\ln(e^{\ln(a)}) = \ln(a)$. So $e^{\ln(a)} = a$.

Good luck!